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Superposition States Through Correlations of the Second Kind

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We introduce model systems, embedded in R^2, for joint systems consisting of two entangled spin-1/2 entities in superposition states: one in a symmetric superposition state, one in an antisymmetric superposition state. We study these model systems and compare them with a model system for a spin-1 quantum entity. All this leads to a new way of looking at the tensor product of Hilbert spaces within the context of quantum mechanics, and thus also to a new approach for the description of joint systems in quantum mechanics.

1. INTRODUCTION

Aerts (1981, 1994) showed that the joint system of two separated quantum entities cannot be described as the projection lattice of the tensor product [for more details on this problem, see Aerts (1981), Piron (1990), and Van Fraasen (1991)]. As a consequence, alternative descriptions of joint systems should be reconsidered. Aerts (1991) presents a model system for an Aspect-like experiment on a quantum system in a singlet state. In this model system, he introduces the concept of *correlations of the second kind*, i.e., correlations created during the measurement process.

In this paper we introduce two models, embedded in R^3, for joint systems consisting of two entangled spin-1/2 entities in a superposition state: one in a symmetric superposition and one in an antisymmetric superposition (this model system for an entity in an antisymmetric superposition state is very much related to Aerts' model system for an Aspect-like experiment). Of course, since we only consider spin-1/2 entities, due to the fermionic superselection rule, the symmetric superposition has no real meaning within the

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context of quantum mechanics. Still, from a structural point of view, this model creates new insights into how states described in a tensor product of two Hilbert spaces can be realized in a mechanistic way. These structural aspects get more interesting when we compare these models with some models for other superposition states and with a model for a spin-1 entity in which we encounter again a joint system of two entangled spin-1/2 quantum entities on which we introduce correlations of the second kind [this model system is introduced in Coecke (1995a)].

We also mention that these correlations of the second kind introduce no new probabilistic aspects: the probabilistic nature of the joint system is due to the probabilistic nature of the composing entities. Thus, if the entities are classical mechanistic, then the joint system is classical mechanistic too, and, since it is possible to build classical mechanistic models for spin-1/2 entities (see, for example, Aerts, 1991; Coecke, 1995b), the models that we present in this paper prove the existence of classical mechanistic models for spin-1 entities and for joint systems in symmetric or antisymmetric states. Nonetheless, the main aim of this paper is structural characterization of new states that occur due to the tensor product, independent of a possible explanation of the origin of the quantum probabilities.

In Section 2 we present a representation of a spin-1/2 quantum entity on the Poincaré sphere which we need in all the models that we introduce in this paper. In Section 3 we introduce the model system for an entity in a symmetric or an antisymmetric superposition state. In Section 4 we introduce a similar model system for the other superposition states, and in Section 5 we briefly describe the model system for a spin-1 quantum entity, as introduced in Coecke (1995a). In Section 6 we study these model systems in more detail and present an alternative way of looking at the states encountered in quantum mechanics, which are described in tensor products of Hilbert spaces.

2. REPRESENTATION OF A SPIN-1/2 QUANTUM ENTITY

In this section, we represent the states of a spin-1/2 quantum entity on a sphere S. Consider a setup with two Stern–Gerlach apparata: one in which we prepare the entity in a certain state, and a second one in which we measure. Clearly, the transition probabilities depend only on the relative position of the two apparata. Thus, we can represent the measurement by the Euler angles α, β, γ (Fig. 1).

We denote a measurement characterized by α, β, γ as $e_{\alpha,\beta,\gamma}$. If the initial state corresponds to a spin quantum number $s = +1/2$, we denote it as p_+^0, and if it corresponds to a spin quantum number $s = -1/2$, we denote it as p_-^0. We represent p_+^0 by the vector $\psi_+^0 = (1, 0) \in C^2$ and p_-^0 by $\psi_-^0 = (0, 1) \in C^2$. The eigenstates corresponding to a measurement $e_{\alpha,\beta,\gamma}$ are the same

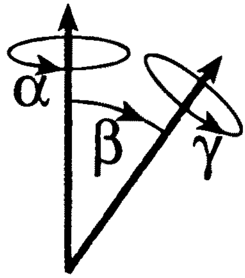

Fig. 1. The angle α represents a first rotation around the direction of the magnetic field. This rotation determines the direction in which we turn with an angle β. Finally, we rotate again around the direction of the magnetic field (represented by the angle γ).

as the ones we obtain when we rotate the initial states by an active rotation characterized by the Euler angles α, β, γ. This active rotation is represented by a unitary operator acting on C^2 that corresponds to the following matrix (Wigner, 1959):

$$M_{\alpha,\beta,\gamma} = \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix} \tag{1}$$

Thus, for the measurement $e_{\alpha,\beta,\gamma}$ we have a set of eigenstates represented by the following vectors:

$$\psi_+^{\alpha,\beta,\gamma} = M_{\alpha,\beta,\gamma}\psi_+^0 = \left(e^{-i\alpha/2}\cos\frac{\beta}{2}, e^{i\alpha/2}\sin\frac{\beta}{2}\right)e^{-i\gamma/2} \tag{2}$$

$$\psi_-^{\alpha,\beta,\gamma} = M_{\alpha,\beta,\gamma}\psi_-^0 = \left(-e^{-i\alpha/2}\sin\frac{\beta}{2}, e^{i\alpha/2}\cos\frac{\beta}{2}\right)e^{i\gamma/2} \tag{3}$$

The vectors in equations (2) and (3) that correspond to different values of γ (for fixed α and β) represent the same states. As a consequence, we omit the superscript γ in the notations for the vectors and the measurements. We represent the states corresponding to the vectors in equations (2) and (3), respectively, by $p_+^{\alpha,\beta}$ and $p_-^{\alpha,\beta}$. Thus we have $p_+^{0,0} = p_+^0$ and $p_-^{0,0} = p_-^0$. We also have

$$p_+^{\alpha,\beta} = p_-^{\alpha+\pi,\pi-\beta} \tag{4}$$

We denote the probability to obtain a state $p_+^{\alpha,\beta}$ in a measurement $e_{\alpha,\beta}$ on an entity in a state p_+^0 as $P_{+,+}^{\alpha,\beta}$, and the probability to obtain $p_-^{\alpha,\beta}$ in a measurement

$e_{\alpha,\beta}$ on an entity in a state p_+^0 as $P_{+,-}^{\alpha,\beta}$. Analogously we define $P_{-,+}^{\alpha,\beta}$ and $P_{-,-}^{\alpha,\beta}$. We have

$$P_{+,+}^{\alpha,\beta} = |\langle\psi_+^0|\psi_+^{\alpha,\beta}\rangle|^2 = \cos^2\frac{\beta}{2} = \frac{1+\cos\beta}{2}$$

$$P_{-,-}^{\alpha,\beta} = \frac{1+\cos\beta}{2}$$

$$P_{+,-}^{\alpha,\beta} = P_{-,+}^{\alpha,\beta} = \sin^2\frac{\beta}{2} = \frac{1-\cos\beta}{2}$$

The set of states of a spin-1/2 entity is given by [see equation (4)]

$$\Sigma_{1/2} = \{p_+^{\alpha,\beta} \,|\, \alpha \in [0, 2\pi], \beta \in [0, \pi]\} \tag{5}$$

Let S be a unit sphere in $\mathbb{R}^3$ with its center at the origin. We represent every state $p_+^{\alpha,\beta} \in \Sigma_{1/2}$ by the point in S with coordinates ($\cos\alpha \sin\beta$, $\sin\alpha \sin\beta$, $\cos\beta$). It is clear (as a consequence of the definition of the Euler angles) that the representation of $\Sigma_{1/2}$ in S is one-to-one and onto.

3. A MODEL SYSTEM FOR SUPERPOSITION STATES

In this section, we present two model systems for measurements on a joint system consisting of two spin-1/2 quantum entities in a superposition state [for more details on this kind of measurement see Aerts (1991)]:

- Before the measurement, we have a joint system consisting of two entangled spin-1/2 entities (denoted as $\mathcal{S}_1$ and $\mathcal{S}_2$) in a superposition state.
- Then we perform a measurement that consists in performing one measurement on each of both spin-1/2 entities.
- After the measurement, we have a joint system consisting of the two separated spin-1/2 entities $\mathcal{S}_1$ and $\mathcal{S}_2$.

First we briefly repeat how such a measurement is described in ordinary quantum mechanics. The measurements consist in the performance of two spin-1/2 measurements on the two spin-1/2 entities of a joint system. If the measurement on $\mathcal{S}_1$ is $e_{\alpha,\beta}$ and if the measurement on $\mathcal{S}_2$ is $e_{\alpha',\beta'}$, we denote the measurement on the joint system as $e_{\alpha,\beta,\alpha',\beta'}$ (we use the notations introduced in Section 2, and again one easily verifies that we are allowed to omit the index γ). The eigenstates of $e_{\alpha,\beta}$ are represented by the vectors $\psi_+^{\alpha,\beta}$ and $\psi_-^{\alpha,\beta}$ in $\mathbb{C}^2$ and the eigenstates of $e_{\alpha',\beta'}$ by $\psi_+^{\alpha',\beta'}$ and $\psi_-^{\alpha',\beta'}$. Thus, according to orthodox quantum mechanics, the possible eigenstates of $e_{\alpha,\beta,\alpha',\beta'}$, denoted by, $p_{+,+}^{\alpha,\beta,\alpha',\beta'}$, $p_{+,-}^{\alpha,\beta,\alpha',\beta'}$, $p_{-,+}^{\alpha,\beta,\alpha',\beta'}$, and $p_{-,-}^{\alpha,\beta,\alpha',\beta'}$, are represented by $\psi_+^{\alpha,\beta} \otimes$

$\psi_+^{\alpha',\beta'}$, $\psi_-^{\alpha,\beta} \otimes \psi_+^{\alpha',\beta'}$, $\psi_+^{\alpha,\beta} \otimes \psi_-^{\alpha',\beta'}$, and $\psi_-^{\alpha,\beta} \otimes \psi_-^{\alpha',\beta'}$, which are all vectors in $C^2 \otimes C^2 = C^4$. The two superposition states that we consider as initial states are an antisymmetric and a symmetric linear combination of product states, denoted as p_1 and p_2. They are represented by the following vectors in C^4 (again we use notations of Section 2):

$$\psi_1 = \frac{1}{\sqrt{2}} (\psi_+^0 \otimes \psi_-^0 - \psi_-^0 \otimes \psi_+^0) \tag{6}$$

$$\psi_2 = \frac{1}{\sqrt{2}} (\psi_+^0 \otimes \psi_-^0 + \psi_-^0 \otimes \psi_+^0) \tag{7}$$

For an initial state p_1, we denote the probability to obtain $p_{+,+}^{\alpha,\beta,\alpha',\beta'}$ in the measurement $e_{\alpha,\beta,\alpha',\beta'}$ as $P_{1,+,+}^{\alpha,\beta,\alpha',\beta'}$, and for an initial state p_2 we denote it as $P_{2,+,+}^{\alpha,\beta,\alpha',\beta'}$. Analogously we define $P_{1,+,-}^{\alpha,\beta,\alpha',\beta'}$, $P_{1,-,+}^{\alpha,\beta,\alpha',\beta'}$, $P_{1,-,-}^{\alpha,\beta,\alpha',\beta'}$, $P_{2,+,-}^{\alpha,\beta,\alpha',\beta'}$, $P_{2,-,+}^{\alpha,\beta,\alpha',\beta'}$, and $P_{2,-,-}^{\alpha,\beta,\alpha',\beta'}$. We have

$$\begin{aligned}
P_{1,+,+}^{\alpha,\beta,\alpha',\beta'} &= |\langle \psi_1 | \psi_+^{\alpha,\beta} \otimes \psi_+^{\alpha',\beta'} \rangle|^2 \\
&= \left| \frac{1}{\sqrt{2}} \langle \psi_+^0 \otimes \psi_-^0 | \psi_+^{\alpha,\beta} \otimes \psi_+^{\alpha',\beta'} \rangle - \frac{1}{\sqrt{2}} \langle \psi_-^0 \otimes \psi_+^0 | \psi_+^{\alpha,\beta} \otimes \psi_+^{\alpha',\beta'} \rangle \right|^2 \\
&= \frac{1}{2} |\langle \psi_+^0 | \psi_+^{\alpha,\beta} \rangle \langle \psi_-^0 | \psi_+^{\alpha',\beta'} \rangle - \langle \psi_-^0 | \psi_+^{\alpha,\beta} \rangle \langle \psi_+^0 | \psi_+^{\alpha',\beta'} \rangle|^2 \\
&= \frac{1}{2} \left| e^{-i\alpha/2} \cos \frac{\beta}{2} e^{i\alpha'/2} \sin \frac{\beta'}{2} - e^{i\alpha/2} \sin \frac{\beta}{2} e^{-i\alpha'/2} \cos \frac{\beta'}{2} \right|^2 \\
&= \frac{1}{2} \left[\cos^2 \frac{\beta}{2} \sin^2 \frac{\beta'}{2} - 2 \cos(\alpha - \alpha') \cos \frac{\beta}{2} \cos \frac{\beta'}{2} \sin \frac{\beta}{2} \sin \frac{\beta'}{2} \right. \\
&\qquad \left. + \sin^2 \frac{\beta}{2} \cos^2 \frac{\beta'}{2} \right] \\
&= \frac{1}{2} \left[\frac{1 + \cos \beta}{2} \frac{1 - \cos \beta'}{2} - \frac{\cos(\alpha - \alpha') \sin \beta \sin \beta'}{2} \right. \\
&\qquad \left. + \frac{1 - \cos \beta}{2} \frac{1 + \cos \beta'}{2} \right] \\
&= \frac{1}{4} (1 - \cos \beta \cos \beta' - \cos(\alpha - \alpha') \sin \beta \sin \beta') \\
&= \frac{1}{4} (1 - \cos \delta_1) = \frac{1}{2} \sin^2 \frac{\delta_1}{2}
\end{aligned}$$

where we have introduced the following notation:

$$\cos \delta_1 = \cos(\alpha - \alpha') \sin \beta \sin \beta' + \cos \beta \cos \beta' \tag{8}$$

If we also introduce the notation

$$\cos \delta_2 = \cos(\alpha - \alpha') \sin \beta \sin \beta' - \cos \beta \cos \beta' \tag{9}$$

we easily find (along the same lines as the calculation of $P^{\alpha,\beta,\alpha',\beta'}_{1,+,+}$)

$$P^{\alpha,\beta,\alpha',\beta'}_{1,+,+} = P^{\alpha,\beta,\alpha',\beta'}_{1,-,-} = \frac{1}{2} \sin^2 \frac{\delta_1}{2} \tag{10}$$

$$P^{\alpha,\beta,\alpha',\beta'}_{1,+,-} = P^{\alpha,\beta,\alpha',\beta'}_{1,-,+} = \frac{1}{2} \cos^2 \frac{\delta_1}{2} \tag{11}$$

$$P^{\alpha,\beta,\alpha',\beta'}_{2,+,+} = P^{\alpha,\beta,\alpha',\beta'}_{2,-,-} = \frac{1}{2} \cos^2 \frac{\delta_2}{2} \tag{12}$$

$$P^{\alpha,\beta,\alpha',\beta'}_{2,+,-} = P^{\alpha,\beta,\alpha',\beta'}_{2,-,+} = \frac{1}{2} \sin^2 \frac{\delta_2}{2} \tag{13}$$

After the measurement, we have a joint system consisting of two separated spin-1/2 quantum entities, and thus its states can be described in $S \times S$.

In accordance with the representation introduced in the previous section, we represent $p^{\alpha,\beta,\alpha',\beta'}_{+,+}$ by a couple consisting of two points with respective coordinates $(\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta)$ and $(\cos \alpha' \sin \beta', \sin \alpha' \sin \beta', \cos \beta')$, $p^{\alpha,\beta,\alpha',\beta'}_{+,-}$ by a couple of points with respective coordinates $(\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta)$ and $(-\cos \alpha' \sin \beta', -\sin \alpha' \sin \beta', -\cos \beta')$, etc.

Now we introduce a model system with the same description as the above quantum entities. Consider two points v and $-v$ located on a sphere, which give $(0, 0, 1)$ and $(0, 0, -1)$ as coordinates. We introduce the measurement $e_{1,u,u'}$ on this entity in the following way (Fig. 2):

- We consider the system as a joint system of two spin-1/2 quantum entities, one of them in a state p_v, the other in a state p_{-v}.

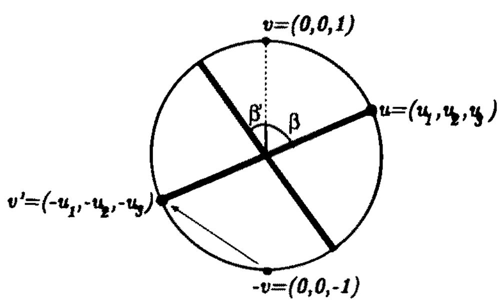

Fig. 2. Illustration of the measurements $e_{u,u'}$ on the joint system in a state described as $\psi_1 = (1/\sqrt{2})(\psi^0_+ \otimes \psi^0_- - \psi^0_- \otimes \psi^0_+)$.

- On one of these two entities, we perform a measurement e_u (we consider measurements on spin-1/2 quantum entities as they are described in Section 2). Let $u = (u_1, u_2, u_3)$. If, as a consequence of this measurement, we obtain p_u, then the state of the other entity changes to $p_{v'}$, where $v' = (-u_1, -u_2, -u_3)$; if we obtain p_{-u}, the state changes to $p_{-v'}$.
- We perform the measurement $e_{u'}$ on the other entity (the order of e_u and $e_{u'}$ makes no difference for the probabilities of the different outcomes).

We introduce the measurement $e_{2,u,u'}$ in the same way as $e_{1,u,u'}$, with one difference: we replace $v' = (-u_1, -u_2, -u_3)$ by $v' = (u_1, u_2, -u_3)$ (Fig. 3).

Denote by β the angle between the vectors u and v, by β' the angle between the vectors u' and v, by δ_1 the angle between the vectors u and u', and by δ_2 the angle between the vectors u and $(u_1, u_2, -u_3)$. We can calculate the probabilities to obtain the different possible outcomes for the measurement $e_{1,u,u'}$. As the probability to become a state $p_{u,u'}$ in the measurement $e_{1,u,u'}$ we have

$$P_{1,v,v',u,u'} = \frac{1}{2}\frac{1+\cos\beta}{2}\frac{1-\cos\delta_1}{2} + \frac{1}{2}\frac{1-\cos\beta}{2}\frac{1-\cos\delta_1}{2}$$

$$= \frac{1}{4}(1-\cos\delta_1) = \frac{1}{2}\sin^2\frac{\delta_1}{2}$$

$$P_{1,v,v',u',u} = \frac{1}{2}\frac{1+\cos\beta'}{2}\frac{1-\cos\delta_1}{2} + \frac{1}{2}\frac{1-\cos\beta'}{2}\frac{1-\cos\delta_1}{2}$$

$$= \frac{1}{2}\sin^2\frac{\delta_1}{2}$$

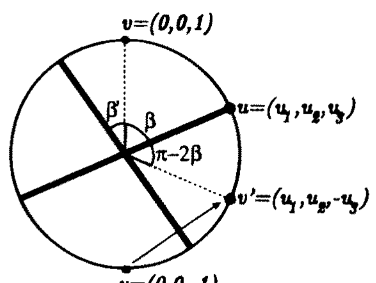

Fig. 3. Illustration of the measurements $e_{u,u'}$ on the joint system in a state described as $\psi_2 = (1/\sqrt{2})(\psi_+^0 \otimes \psi_-^0 + \psi_-^0 \otimes \psi_+^0)$.

and thus, as already announced, the order of e_u and $e_{u'}$ has no importance. For the probability to become a state $p_{u,u'}$ in the measurement $e_{2,u,u'}$ we have

$$P_{2,v,v',u,u'} = \frac{1}{2}\frac{1+\cos\beta}{2}\frac{1+\cos\delta_2}{2} + \frac{1}{2}\frac{1-\cos\beta}{2}\frac{1+\cos\delta_2}{2}$$

$$= \frac{1}{4}(1+\cos\delta_2) = \frac{1}{2}\cos^2\frac{\delta_2}{2}$$

Analogously, we find the other probabilities. If we parametrize the coordinates of u as $(\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta)$ and of u' as $(\cos\alpha'\sin\beta', \sin\alpha'\sin\beta', \cos\beta')$, we find

$$\begin{aligned}\cos\delta_1 &= (\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta)\\ &\quad\times(\cos\alpha'\sin\beta', \sin\alpha'\sin\beta', \cos\beta')\\ &= \cos(\alpha-\alpha')\sin\beta\sin\beta' + \cos\beta\cos\beta'\\ \cos\delta_2 &= (\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta)\\ &\quad\times(\cos\alpha'\sin\beta', \sin\alpha'\sin\beta', -\cos\beta')\\ &= \cos(\alpha-\alpha')\sin\beta\sin\beta' - \cos\beta\cos\beta'\end{aligned}$$

Thus, this justifies the double choice of these notations [according to equations (16) and (17)]. If we compare the expressions for $P_{1,v,v',u,u'}$, $P_{1,v,v',-u,u'}$, $P_{1,v,v',u,-u'}$, $P_{1,v,v',-u,-u'}$, $P_{2,v,v',u,u'}$, $P_{2,v,v',-u,u'}$, $P_{2,v,v',u,-u'}$, and $P_{2,v,v',-u,-u'}$ with equations (10)–(13), we find that the measurements $e_{1,u,u'}$ are representations of measurements on a quantum entity in a antisymmetric superposition state p_1, and that the measurements $e_{2,u,u'}$ are representations of measurements on a quantum entity in a symmetric superposition state p_2. We remark that all this means that the superposition states ψ_1 and ψ_2 can also be represented in $S \times S$, both by the same two points (0, 0, 1) and (0, 0, −1), if we introduce correlations of the second kind.

4. MODEL SYSTEMS FOR SOME OTHER SUPERPOSITION STATES

In this section we present models of two other superposition states. Clearly, from a quantum mechanical point of view, they have no real meaning. But, as already mentioned in the introduction, from a structural point of view, they are worth noticing. We consider the states p_3 and p_4 represented by the following vectors:

$$\psi_3 = \frac{1}{\sqrt{2}} (\psi_+^0 \otimes \psi_+^0 - \psi_-^0 \otimes \psi_-^0) \tag{14}$$

$$\psi_4 = \frac{1}{\sqrt{2}} (\psi_+^0 \otimes \psi_+^0 + \psi_-^0 \otimes \psi_-^0) \tag{15}$$

We denote the probabilities to obtain certain outcomes in an analogous way as we did in the previous section. If we introduce the notations

$$\cos \delta_3 = \cos(\alpha + \alpha') \sin \beta \sin \beta' - \cos \beta \cos \beta' \tag{16}$$

$$\cos \delta_4 = \cos(\alpha + \alpha') \sin \beta \sin \beta' + \cos \beta \cos \beta' \tag{17}$$

we easily find (along the same lines as the calculation in the previous section)

$$P_{3,+,+}^{\alpha,\beta,\alpha',\beta'} = P_{3,-,-}^{\alpha,\beta,\alpha',\beta'} = \frac{1}{2} \sin^2 \frac{\delta_3}{2} \tag{18}$$

$$P_{3,+,-}^{\alpha,\beta,\alpha',\beta'} = P_{3,-,+}^{\alpha,\beta,\alpha',\beta'} = \frac{1}{2} \cos^2 \frac{\delta_3}{2} \tag{19}$$

$$P_{4,+,+}^{\alpha,\beta,\alpha',\beta'} = P_{4,-,-}^{\alpha,\beta,\alpha',\beta'} = \frac{1}{2} \cos^2 \frac{\delta_4}{2} \tag{20}$$

$$P_{4,+,-}^{\alpha,\beta,\alpha',\beta'} = P_{4,-,+}^{\alpha,\beta,\alpha',\beta'} = \frac{1}{2} \sin^2 \frac{\delta_4}{2} \tag{21}$$

Again we consider two points v and $-v$ located on a sphere with respective coordinates (0, 0, 1) and (0, 0, -1). We introduce the measurement $e_{3,u,u'}$ in the same way as $e_{1,u,u'}$ by replacing $v' = (-u_1, -u_2, -u_3)$ by $v' = (u_1, -u_2, -u_3)$. We introduce $e_{4,u,u'}$ by replacing $v' = (-u_1, -u_2, -u_3)$ by $v' = (u_1, -u_2, u_3)$. Denote by δ_3 the angle between the vectors u and $(-u_1, u_2, u_3)$ and by δ_4 the angle between the vectors u and $(u_1, -u_2, u_3)$. We have

$$P_{3,v,v',u,u'} = \frac{1}{2} \frac{1 + \cos \beta}{2} \frac{1 - \cos \delta_3}{2} + \frac{1}{2} \frac{1 - \cos \beta}{2} \frac{1 - \cos \delta_3}{2} = \frac{1}{2} \sin^2 \frac{\delta_3}{2}$$

$$P_{4,v,v',u,u'} = \frac{1}{2} \frac{1 + \cos \beta}{2} \frac{1 + \cos \delta_4}{2} + \frac{1}{2} \frac{1 - \cos \beta}{2} \frac{1 + \cos \delta_4}{2} = \frac{1}{2} \cos^2 \frac{\delta_4}{2}$$

and analogously, we find the probabilities for the other outcomes. If we parametrize the coordinates of u as $(\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta)$ and of u' as $(\cos \alpha' \sin \beta', \sin \alpha' \sin \beta', \cos \beta')$, we find

$$\begin{aligned}\cos \delta_3 &= (\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta) \\ &\quad \times (\cos \alpha' \sin \beta', -\sin \alpha' \sin \beta', -\cos \beta') \\ &= \cos(\alpha + \alpha') \sin \beta \sin \beta' - \cos \beta \cos \beta'\end{aligned}$$

$$\cos \delta_4 = (\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta)$$
$$\times (\cos \alpha' \sin \beta', -\sin \alpha' \sin \beta', \cos \beta')$$
$$= \cos(\alpha + \alpha') \sin \beta \sin \beta' + \cos \beta \cos \beta'$$

which completes the proof of the equivalence.

5. A MODEL SYSTEM FOR A SPIN-1 QUANTUM ENTITY

This model system has been introduced in Coecke (1995a,b). Here we only describe the model system. For proofs and details we refer to these two previous publications.

We denote a measurement characterized by the Euler angles α, β, γ as $e_{\alpha,\beta,\gamma}$ (in the sense as in the case of the spin-1/2 entity of Section 2). If the initial state of the entity corresponds to a spin quantum number $s = +1$, we denote it as p_+^0. If $s = 0$, we denote it as p_0^0, and if $s = -1$, as p_-^0. We represent p_+^0 by the vector $\psi_+^0 = (1, 0, 0) \in C^3$, p_0^0 by $\psi_0^0 = (0, 1, 0) \in C^3$, and p_-^0 by $\psi_-^0 = (0, 0, 1) \in C^3$. For the measurement $e_{\alpha,\beta,\gamma}$ denote the eigenstates respectively by $p_+^{\alpha,\beta}$, $p_0^{\alpha,\beta}$, and $p_-^{\alpha,\beta}$. We define the following two subsets of the set of coherent spin states [for more details we refer to Aaberge (1994) and Coecke (1995a)]:

$$\Sigma_+ = \{p_+^{\alpha,\beta} | \alpha \in [0, 2\pi], \beta \in [0, \pi]\} \tag{22}$$

$$\Sigma_0 = \{p_0^{\alpha,\beta} | \alpha \in [0, 2\pi], \beta \in [0, \pi]\} \tag{23}$$

The set of all coherent spin-1 states is $\Sigma_1 = \Sigma_+ \cap \Sigma_0$. If $p_+^{\alpha,\beta} \in \Sigma_+$, we represent this state by two identical points in S (one point of $S \times S$) with coordinates $(\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta)$. If $p_0^{\alpha,\beta} \in \Sigma_0$, we represent this state by two points in S with respective coordinates $(\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta)$ and $(-\cos \alpha \sin \beta, -\sin \alpha \sin \beta, -\cos \beta)$.

Suppose that the states of the model system can be represented by the same subset of $S \times S$ as we used in the previous section for the representation of the coherent states of a spin-1 quantum entity. A state represented as two identical points on the sphere with coordinates v will be denoted as $p_{v,v}$, and a state represented as two diametrically opposite points with coordinates v and $-v$ as $p_{v,-v}$. We choose a set of coordinates such that $v = (0, 0, 1)$. We define a measurement e_u on the system in a state $p_{v,v}$ in the following way:

- We consider the system as a joint system of two separated spin-1/2 quantum entities in a state p_v.
- On both entities perform a measurement with eigenstates p_u and p_{-u}.

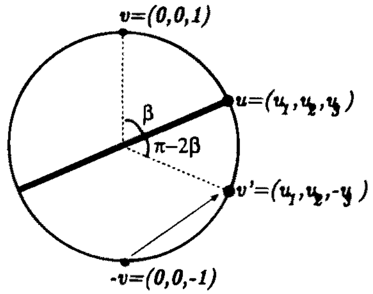

Fig. 4. Illustration of the e_u measurements on the model system in a state $p_{v,-v}$. In this illustration, the black dots represent the position of $\mathcal{S}_1$.

We define the measurement e_u on the system in a state $p_{-v,v}$ in the following way (Fig. 4):

- We consider the system as a joint system of two spin-1/2 quantum entities that are entangled, one of them in a state p_v, the other in a state p_{-v}.
- On one of these two entities, which we denote as $\mathcal{S}_1$, we perform a measurement with eigenstates p_u and p_{-u}. Let $u = (u_1, u_2, u_3)$. If, as a consequence of this measurement, we obtain a state p_u for $\mathcal{S}_1$, then the state of the other entity (denoted as $\mathcal{S}_2$) changes to $p_{v'}$, where $v' = (u_1, u_2, -u_3)$. If we obtain a state p_{-u} for $\mathcal{S}_1$, then the state of $\mathcal{S}_2$ changes to $p_{v'}$.
- We perform a measurement with eigenstates p_u and p_{-u} on $\mathcal{S}_2$.

There are three possible outcome states for e_u: $p_{u,u}$, $p_{-u,-u}$, or $p_{-u,u}$. Denote by β the angle between the vectors u and v. We can identify the state $p_{v,v}$ with the state $p_+^0 \in \Sigma_1$ of a spin-1 quantum entity, the state $p_{v,-v}$ with the state $p_0^0 \in \Sigma_1$, the state $p_{u,u}$ with the state $p_+^{\alpha,\beta} \in \Sigma_1$, and the state $p_{u,-u}$ with the state $p_0^{\alpha,\beta} \in \Sigma_1$. As proved in Coecke (1995a), we find the same transition probabilities for the model system as for a spin-1 quantum entity. Thus, as in the case of the measurements on entities in superposition states, all states are represented in $S \times S$. Moreover, since the collections of possible initial and outcome states are the same, we can consider consecutive measurements within this model system.

6. DISCUSSION

Aerts (1991) shows with a model system that the specific probabilistic structure that arises during measurements on entities in a singlet state can be understood as due to the presence of correlations of the second kind in these measurements. In all the model systems introduced in this paper, we can also identify these correlations of the second kind: the specific kind of correlation that induces a transition of the points of $\mathcal{S}_2$ depends on the outcome of the measurement on $\mathcal{S}_1$; since this outcome does not exist before the measurement is actually performed, the correlation is created during the measurement.

The singlet state is a new state which is not contained in $\mathcal{H}(\mathbb{C}^2) \times \mathcal{H}(\mathbb{C}^2)$ [$\mathcal{H}(X)$ are the rays of X], but in $\mathcal{H}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ [we remark that $\mathcal{H}(\mathbb{C}^2) \times \mathcal{H}(\mathbb{C}^2)$ are the product states of $\mathcal{H}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, i.e., the states that correspond with two separated spin-1/2 quantum entities]. In the model system introduced by Aerts (1991), the singlet state is no longer represented in $\mathcal{H}(\mathbb{C}^2) \times \mathcal{H}(\mathbb{C}^2)$, i.e., no longer represented as a couple of points on a sphere, but it is represented as two identical points in the middle of the sphere (Aerts, 1991). Still, as we showed in this paper, if we introduce correlations of the second kind in the measurements, this new state (and also the other superposition states that we considered), which does not correspond to a product state in $\mathcal{H}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, can be "represented" as a product state. Moreover, the same counts for the spin-1 model. The set of all coherent spin-1 states cannot be embedded in $\mathcal{H}(\mathbb{C}^2) \times \mathcal{H}(\mathbb{C}^2)$, but is a subset of $\mathcal{H}(\mathbb{C}^3)$. Since $\mathcal{H}(\mathbb{C}^3)$ is a subspace of $\mathcal{H}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, we find new states in $\mathcal{H}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ that are not contained in $\mathcal{H}(\mathbb{C}^2) \times \mathcal{H}(\mathbb{C}^2)$, and this is again due to the presence of correlations of the second kind in the measurements. Thus, from a mathematical point of view, one sees that instead of "extending" the state space of a system by applying larger representation spaces for the description of joint systems,[2] one can as well introduce correlations that are created during the measurement. From a physical point of view, all this leads to a distinction between the description of the actual state of a joint system in Piron's sense[3] (which leads to a tensor product representation), and a description in which we focus on the possible states of the individual entities in the joint system and which requires the introduction of correlations of the second kind to express the entanglement of the entities of which the joint system consists. Thus, we can introduce a different approach toward the description of joint systems: *if a joint system consists of two separated entities, the states are represented by the Cartesian product of the state spaces of these entities; if due to the interaction between the measurement apparatus and the joint system, correlations of the second kind are created, new states occur, i.e., new kinds of behavior of the entity during the measurement occur.* But, as already mentioned above, these new states (i.e., these new ways of behavior of the entity during the measurement) are still "represented" within the Cartesian product.

[2] In the case of quantum mechanics this means the introduction of the tensor product as a tool for the description of joint systems.

[3] A description in which we incorporate every possible potential behavior of the entity during every possible measurement. For more details see Piron (1976).

Of course, the argumentation put forward in this paper is of a somewhat metaphorical nature, and thus harder mathematical evidence in support of the above ideas is required. Nonetheless, we think that the examples introduced in this paper explain these ideas in a more transparent way than a purely mathematical treatment would do. A paper on a purely mathematical treatment is in preparation.

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